

# STRAIGHTENING RULE FOR AN $m'$ -TRUNCATED POLYNOMIAL RING

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**ABSTRACT.** We consider a certain quotient of a polynomial ring categorified by both the isomorphic Green rings of symmetric group and Schur algebra generated by the signed Young permutation modules and mixed powers respectively. They have bases parametrised by pairs of partitions whose second partitions are multiple of the odd prime  $p$  the characteristic of the underlying field. We provide an explicit formula rewriting a signed Young permutation module (respectively, mixed power) in terms of signed Young permutation modules (respectively, mixed powers) labelled by those pair of partitions. As a result, the combinatorial number, for each partition, the number of compositions which can be rearranged to the partition and whose all partial sums are not divisible by  $p$  arises naturally.

## 1. INTRODUCTION

The ring of symmetric functions plays an important role in both combinatorics and representation theory of the symmetric groups (see [13, §I.7] and [12]). In the characteristic zero case, the characteristic map is an isometric isomorphism between the ring of symmetric functions and the Green ring for symmetric group where Schur functions correspond to Specht modules, the complete and elementary symmetric functions correspond to the trivial and signature representations for the symmetric groups such that the multiplication of two symmetric functions correspond to the induction of the outer tensor product of the two respective modules. In the positive characteristic case, the ring of symmetric functions is isomorphic to the Green ring of the symmetric group generated by Young permutation modules.

Let  $k$  be a field of odd characteristic  $p$ . In [4], Donkin showed that the isomorphism classes of the signed Young permutation modules

$$M(\alpha|\beta) = \text{Ind}_{\mathfrak{S}_\alpha \times \mathfrak{S}_\beta}^{\mathfrak{S}_n}(k(\alpha) \boxtimes \text{sgn}(\beta)),$$

where  $(\alpha|\beta)$  is a pair of compositions such that the sizes of  $\alpha$  and  $\beta$  sum up to  $n$  and  $k(\alpha), \text{sgn}(\beta)$  are the trivial and sign  $k\mathfrak{S}_\alpha$ - and  $k\mathfrak{S}_\beta$ -modules respectively, are parametrised by the set  $\mathcal{P}_p^2(n)$  consisting of pairs of partitions of the form  $(\lambda|p\mu)$  such that  $|\lambda| + p|\mu| = n$ . Furthermore,  $M(\lambda|p\mu)$  has a distinguished indecomposable summand the signed Young module  $Y(\lambda|p\mu)$  of multiplicity one such that any other summand of  $M(\lambda|p\mu)$  is of the form  $Y(\delta|p\xi)$  for some  $(\delta|p\xi) \triangleright (\lambda|p\mu)$ , here,  $\triangleright$  is certain dominance order on  $\mathcal{P}_p^2(n)$ . In other words, the Green ring of the symmetric

group generated by signed Young permutation modules, denoted by  $\mathcal{Y}(\mathfrak{S})$ , has  $\mathbb{Z}$ -bases  $\mathcal{C} = \{[M(\lambda|p\mu)] : (\lambda|p\mu) \in \mathcal{P}_p^2\}$  and  $\{[Y(\lambda|p\mu)] : (\lambda|p\mu) \in \mathcal{P}_p^2\}$  where  $\mathcal{P}_p^2 = \bigcup_{n \in \mathbb{N}_0} \mathcal{P}_p^2(n)$ .

Let  $E$  be the natural module for the Schur algebra  $S(\infty, 1)$ . The mixed powers are defined as

$$K^{\alpha|\beta}E = S^\alpha E \otimes \bigwedge^\beta E.$$

For each  $(\lambda|p\mu) \in \mathcal{P}_p^2$ , the mixed power  $K^{\lambda|p\mu}E$  has a distinguished indecomposable summand the listing module  $\text{List}^{\lambda|p\mu}E$  of multiplicity one such that any other summand of  $K^{\lambda|p\mu}E$  is of the form  $\text{List}^{\delta|p\xi}E$  for some  $(\delta|p\xi) \triangleright (\lambda|p\mu)$ . In other words, the Green ring of the Schur algebra generated by mixed powers, denoted by  $\mathcal{L}(S)$ , has  $\mathbb{Z}$ -bases  $\mathcal{D} = \{[K^{\lambda|p\mu}E] : (\lambda|p\mu) \in \mathcal{P}_p^2\}$  and  $\{[\text{List}^{\lambda|p\mu}E] : (\lambda|p\mu) \in \mathcal{P}_p^2\}$ . In fact,  $\mathcal{L}(S)$  is isomorphic to  $\mathcal{Y}(\mathfrak{S})$  induced by the Schur functor where  $K^{\alpha|\beta}E$  and  $\text{List}^{\lambda|p\mu}E$  are mapped to  $M(\alpha|\beta)$  and  $Y(\lambda|p\mu)$  respectively.

In the classical case, the class of non-isomorphic direct summands of Young permutation  $k\mathfrak{S}_n$ -modules are known as Young modules and labelled by the set of partitions of  $n$ . Determining the multiplicity of a Young module as a direct summand of a Young permutation module is an open problem (see [3, 5, 7, 9, 11]). The numbers are known as the  $p$ -Kostka numbers. The signed  $p$ -Kostka numbers are generalisation of the  $p$ -Kostka numbers which are the multiplicities of signed Young modules as direct summands of signed Young permutation modules (see [6]). While writing a signed Young permutation module as  $\mathbb{Z}$ -linear combination of the signed Young modules in the Green ring is an open problem, in this article, we present an explicit formula writing a signed Young permutation module (respectively, a mixed power) in terms of the basis  $\mathcal{C}$  (respectively,  $\mathcal{D}$ ). The proof relies on the categorification theorem of by certain quotient  $\Gamma^{(p)}$  of a polynomial ring by both  $\mathcal{Y}(\mathfrak{S})$  and  $\mathcal{L}(S)$  proved by Donkin in [4]. Along the way, the combinatorial number, for each partition  $\lambda$ , the number of compositions  $\delta$  such that  $\delta$  can be rearranged to  $\lambda$  and all partial sums of  $\delta$  are not divisible by  $p$  arises naturally. The numbers appear in the coefficients of rewriting the sign representations (or the exterior powers) in terms of the basis  $\mathcal{C}$  (or  $\mathcal{D}$ ).

The article is organised as follows. In the next section, we recall the symmetric functions, certain quotient  $\Gamma^{(m)}$  of a polynomial ring, where  $m$  is a positive integer, the Green rings  $\mathcal{Y}(\mathfrak{S})$  and  $\mathcal{L}(S)$ . In Section 3, we study some properties of the ring  $\Gamma^{(m)}$ . In Section 4, we prove the main result Theorem 4.2 and deduce the formulae of writing a signed Young permutation module and a mixed power in terms of the bases  $\mathcal{C}$  and  $\mathcal{D}$  respectively. As a consequence of the formulae, for any pair of composition  $(\alpha|\beta)$ , we deduce the ‘canonical’ summand of the signed Young permutation module  $M(\alpha|\beta)$  (respectively, the mixed power  $K^{\alpha|\beta}E$ ).

## 2. PRELIMINARIES

In this section, we fix the notation we shall require throughout in the article and introduce the background material. The standard references are [4, 8, 10, 13].

Let  $\mathbb{Z}$  be the set of integers, let  $\mathbb{N}$  be the set of positive integers and let  $\mathbb{N}_0$  be the set of non-negative integers. For a finite set  $A$ , let  $\mathfrak{S}_A$  denote the symmetric group acting on the set  $A$ . For each  $n \in \mathbb{N}_0$ , let

$$[n] = \{i \in \mathbb{N} : 1 \leq i \leq n\}$$

and  $\mathfrak{S}_n = \mathfrak{S}_{[n]}$ . By convention,  $[0] = \emptyset$  and  $\mathfrak{S}_0$  is the trivial group.

Let  $n \in \mathbb{N}_0$ . A composition  $\lambda$  of  $n$  is a sequence of positive integers  $(\lambda_1, \dots, \lambda_r)$  such that  $\sum_{i=1}^r \lambda_i = n$ . In this case, we write  $\ell(\lambda) = r$  and  $|\lambda| = n$ . By convention, the unique composition of 0 is denoted as  $\emptyset$  and  $\ell(\emptyset) = 0$ . The set of all compositions of  $n$  is denoted by  $\mathcal{C}(n)$  and we write  $\mathcal{C} = \bigcup_{n \in \mathbb{N}_0} \mathcal{C}(n)$ . The composition  $\lambda$  is called a partition if  $\lambda_1 \geq \dots \geq \lambda_r$ . We write  $\mathcal{P}(n)$  for the set of partitions of  $n$  and  $\mathcal{P} = \bigcup_{n \in \mathbb{N}_0} \mathcal{P}(n)$  for the set of all partitions.

The concatenation of two compositions  $\alpha$  and  $\beta$  is the composition

$$\alpha \# \beta = (\alpha_1, \dots, \alpha_{\ell(\alpha)}, \beta_1, \dots, \beta_{\ell(\beta)}).$$

By convention,  $\alpha \# \emptyset = \alpha$  and, similarly,  $\emptyset \# \beta = \beta$ . Let  $m \in \mathbb{N}$  and  $\mu = (\mu_1, \dots, \mu_s)$  be a composition. We write  $m\mu$  for the composition  $(m\mu_1, \dots, m\mu_s)$  of  $m|\mu|$ .

The set of all pairs  $(\alpha|\beta)$  of compositions of  $n$ , i.e.,  $\alpha, \beta$  are compositions and  $|\alpha| + |\beta| = n$ , is denoted by  $\mathcal{C}^2(n)$ . We write  $\mathcal{C}^2 = \bigcup_{n \in \mathbb{N}_0} \mathcal{C}^2(n)$ . Similarly, we write  $\mathcal{P}^2(n)$  for the set of pairs of partitions of  $n$  and  $\mathcal{P}^2 = \bigcup_{n \in \mathbb{N}_0} \mathcal{P}^2(n)$ . For  $m \in \mathbb{N}$ , the subset of  $\mathcal{P}^2(n)$  consisting of pairs of the form  $(\lambda|m\mu)$  is denoted by  $\mathcal{P}_m^2(n)$ . Furthermore, we write  $\mathcal{P}_m^2 = \bigcup_{n \in \mathbb{N}_0} \mathcal{P}_m^2(n)$ .

Throughout,  $k$  is a field of odd characteristic  $p$ .

**2.1. Symmetric function.** Let  $\mathbb{Z}[[X]]$  be the set of formal power series in the set consisting of the countably infinite commuting variables  $X = \{x_i : i \in \mathbb{N}\}$  with coefficients in  $\mathbb{Z}$  such that  $\deg(x_i) = 1$  for all  $i \in \mathbb{N}$ . For each  $n \in \mathbb{N}$ , the symmetric group  $\mathfrak{S}_n$  acts on  $\mathbb{Z}[[X]]$  by permuting the variables  $x_1, \dots, x_n$  naturally, i.e., for each  $\sigma \in \mathfrak{S}_n$  and  $f(X) \in \mathbb{Z}[[X]]$ , the element  $\sigma \cdot f(X)$  is obtained from  $f(X)$  by replacing each  $x_i$  with  $x_{\sigma(i)}$ . The element  $f(X)$  is called a symmetric function if it is invariant under the actions  $\mathfrak{S}_n$  for all  $n \in \mathbb{N}$ .

The set of symmetric functions with degrees bounded above in the set of variables  $X$  is denoted by  $\Lambda(X)$ . Notice that  $\Lambda(X)$  is a commutative graded ring where  $\Lambda(X) = \bigoplus_{n \in \mathbb{N}_0} \Lambda^n(X)$  and  $\Lambda^n(X)$  consists of homogeneous symmetric functions of degree  $n$ . In the theory of the symmetric function,  $\Lambda(X)$  has different  $\mathbb{Z}$ -bases of special interest. In particular, we are interested in the elementary symmetric functions and complete symmetric functions which we shall now describe.

For any  $n \in \mathbb{N}$ , the  $n$ th elementary symmetric function and the  $n$ th complete symmetric function are

$$e_n(X) = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n},$$

$$h_n(X) = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n},$$

respectively. By convention, let  $e_0(X) = 1 = h_0(X)$  and  $h_n(X) = 0 = e_n(X)$  if  $n < 0$ . For each composition  $\alpha = (\alpha_1, \dots, \alpha_r)$ , let

$$\begin{aligned} e_\alpha(X) &= e_{\alpha_1}(X) \cdots e_{\alpha_r}(X), \\ h_\alpha(X) &= h_{\alpha_1}(X) \cdots h_{\alpha_r}(X). \end{aligned}$$

By convention,  $e_\emptyset(X) = 1 = h_\emptyset(X)$ . Clearly, if  $\alpha$  can be rearranged to a partition  $\lambda$  then  $e_\alpha(X) = e_\lambda(X)$  and  $h_\alpha(X) = h_\lambda(X)$ . The following results are well-known.

**Theorem 2.1.**

- (i) *The ring of symmetric functions  $\Lambda(X)$  is the polynomial ring in  $\{h_n(X) : n \in \mathbb{N}\}$  and  $\{e_n(X) : n \in \mathbb{N}\}$  respectively over  $\mathbb{Z}$ . Furthermore, the subsets  $\{e_\lambda(X) : \lambda \in \mathcal{P}\}$  and  $\{h_\lambda(X) : \lambda \in \mathcal{P}\}$  are  $\mathbb{Z}$ -bases for  $\Lambda(X)$ .*
- (ii) *For each  $d \in \mathbb{N}$ , we have  $\sum_{i=0}^d (-1)^i h_i(X) e_{d-i}(X) = 0$ .*

**2.2. An  $m'$ -truncated polynomial ring.** Let  $X = \{x_i : i \in \mathbb{N}\}$  and  $Y = \{y_j : j \in \mathbb{N}\}$  be two sets of independent countably infinite commuting variables. Clearly,  $\Lambda(X)$  and  $\Lambda(Y)$  are graded subrings of  $\mathbb{Z}[[X \cup Y]]$ . Let  $\langle \Lambda(X), \Lambda(Y) \rangle$  be the subring of  $\mathbb{Z}[[X \cup Y]]$  generated by  $\Lambda(X)$  and  $\Lambda(Y)$ . By Theorem 2.1(i), it is clear that  $\langle \Lambda(X), \Lambda(Y) \rangle$  is a polynomial ring in  $\{h_i(X), e_j(Y) : i, j \in \mathbb{N}\}$  over  $\mathbb{Z}$  and has a  $\mathbb{Z}$ -basis  $\{h_\lambda(X) e_\mu(Y) : (\lambda|\mu) \in \mathcal{P}^2\}$  whose  $n$ th component has  $\mathbb{Z}$ -basis  $\{h_\lambda(X) e_\mu(Y) : (\lambda|\mu) \in \mathcal{P}^2(n)\}$ .

Fix a positive integer  $m$ . We define the quotient

$$\Gamma^{(m)} = \langle \Lambda(X), \Lambda(Y) \rangle / I$$

where  $I$  is the ideal of  $\langle \Lambda(X), \Lambda(Y) \rangle$  generated by  $\sum_{i=0}^d (-1)^i h_i(X) e_{d-i}(Y)$  for every positive integer  $d$  such that  $m \nmid d$ . For each  $i, j \in \mathbb{N}_0$  and  $\alpha, \beta \in \mathcal{C}$ , we denote

$$h_i = h_i(X) + I, \quad e_j = e_j(Y) + I, \quad h_\alpha = h_\alpha(X) + I, \quad e_\beta = e_\beta(Y) + I.$$

We call  $\Gamma^{(m)}$  the ring of  $m'$ -truncated symmetric functions in the variables  $X \cup Y$ .

Since  $I$  is a homogeneous ideal of  $\langle \Lambda(X), \Lambda(Y) \rangle$ , the ring  $\Gamma^{(m)}$  is graded with the  $n$ th component as

$$(\Gamma^{(m)})^n = (\langle \Lambda(X), \Lambda(Y) \rangle^n + I) / I$$

consisting of the  $\mathbb{Z}$ -linear combinations of the  $m'$ -truncated symmetric functions  $h_\alpha e_\beta$  such that  $|\alpha| + |\beta| = n$ , together with the zero element.

**Theorem 2.2** ([4, §4.1(10,11)]). *The ring  $\Gamma^{(m)}$  is the polynomial ring in  $\{h_i, e_{jm} : i, j \in \mathbb{N}\}$  and it has a  $\mathbb{Z}$ -basis*

$$\mathcal{B} = \{h_\lambda e_{m\mu} : (\lambda|m\mu) \in \mathcal{P}_m^2\}.$$

Furthermore, there is a graded involution  $\psi : \Gamma^{(m)} \rightarrow \Gamma^{(m)}$  given by  $\psi(h_i) = e_i$ .

**2.3. Signed Young permutation module.** The general theory of the representation theory of the symmetric group can be found, for example, in [10].

Let  $m, n \in \mathbb{N}_0$ . We denote  $\mathfrak{S}_n^{+m}$  for the symmetric group acting on the set  $\{i + m : i \in [n]\}$  consisting of permutations of the forms  $\tau^{+m}$  where  $\tau \in \mathfrak{S}_n$  such that, for all  $i \in [n]$ ,

$$\tau^{+m}(i + m) = \tau(i) + m.$$

Furthermore, we identify  $\mathfrak{S}_m \times \mathfrak{S}_n$  with the subgroup  $\mathfrak{S}_m \mathfrak{S}_n^{+m}$  of  $\mathfrak{S}_{m+n}$ . For a given composition  $\alpha = (\alpha_1, \dots, \alpha_r)$  of  $n$ , we denote the Young subgroup of  $\mathfrak{S}_n$  with respect to  $\alpha$  by

$$\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \dots \times \mathfrak{S}_{\alpha_r}.$$

For each  $n \in \mathbb{N}_0$ , the set  $\mathcal{P}_p^2(n)$  is partially ordered by the dominance order  $\supseteq$  where  $(\lambda|p\mu) \supseteq (\delta|p\xi)$  if

- (a)  $\sum_{i=1}^{\ell} \lambda_i \geq \sum_{i=1}^{\ell} \delta_i$ , and
- (b)  $|\lambda| + p \sum_{i=1}^{\ell} \mu_i \geq |\delta| + p \sum_{i=1}^{\ell} \xi_i$ ,

for all  $\ell \in \mathbb{N}$ , where  $\lambda_i$  is treated as 0 if  $i > \ell(\lambda)$  and similarly for  $\mu_i$ ,  $\delta_i$  and  $\xi_i$ . In the case when  $(\lambda|p\mu) \supseteq (\delta|p\xi)$  and  $(\lambda|p\mu) \neq (\delta|p\xi)$ , we write  $(\lambda|p\mu) \supset (\delta|p\xi)$ .

Recall that  $k$  is a field of odd characteristic  $p$ . In this article, we denote  $\otimes$  and  $\boxtimes$  for tensor product and outer tensor product of modules over  $k$ .

We write  $k(n)$  and  $\text{sgn}(n)$  for the trivial and signature representations for  $k\mathfrak{S}_n$ . For a composition  $\alpha \in \mathcal{C}(n)$ ,  $k(\alpha)$  and  $\text{sgn}(\alpha)$  are the restrictions of  $k(n)$  and  $\text{sgn}(n)$  respectively to the Young subgroup  $\mathfrak{S}_\alpha$ . Let  $(\alpha|\beta) \in \mathcal{C}^2(n)$ . We define the signed Young permutation  $k\mathfrak{S}_n$ -module  $M(\alpha|\beta)$  with respect to  $(\alpha|\beta)$  as

$$M(\alpha|\beta) = \text{Ind}_{\mathfrak{S}_\alpha \times \mathfrak{S}_\beta}^{\mathfrak{S}_n} (k(\alpha) \boxtimes \text{sgn}(\beta)).$$

Following the work of Donkin [4, §2.3], we know that all non-isomorphic indecomposable summands of the signed Young permutation  $k\mathfrak{S}_n$ -modules are parametrised by the set  $\mathcal{P}_p^2(n)$ . They are called signed Young  $k\mathfrak{S}_n$ -modules and denoted by  $Y(\lambda|p\mu)$  such that, for each  $(\lambda|p\mu) \in \mathcal{P}_p^2(n)$ ,  $Y(\lambda|p\mu)$  is a summand of  $M(\lambda|p\mu)$  with multiplicity one and the remaining summands of  $M(\lambda|p\mu)$  are of the form  $Y(\delta|p\theta)$  where  $(\delta|p\theta) \in \mathcal{P}_p^2(n)$  and  $(\delta|p\theta) \supset (\lambda|p\mu)$ .

Let  $G$  be a group and  $\mathcal{R}(kG)$  be the Green ring (or representation ring) of  $kG$  consisting of formal linear combinations of the isomorphism classes  $[V]$  of finite dimensional indecomposable  $kG$ -modules  $V$  over  $\mathbb{Z}$  such that  $[V] = [W]$  if and only if  $V \cong W$ ,  $[V] + [W] = [V \oplus W]$  and the (inner) product given by  $[V] \cdot [W] = [V \otimes W]$ . The contragredient dual induces an automorphism on  $\mathcal{R}(kG)$  since  $(V \oplus W)^* \cong V^* \oplus W^*$  and  $(V \otimes W)^* \cong V^* \otimes W^*$ .

For each  $n \in \mathbb{N}_0$ , we have the Green ring  $\mathcal{R}(k\mathfrak{S}_n)$  where, by convention,  $\mathcal{R}(k\mathfrak{S}_0) \cong \mathbb{Z}$  has  $\mathbb{Z}$ -basis consisting only the trivial  $k\mathfrak{S}_0$ -module. Let  $V$  and  $W$  be  $k\mathfrak{S}_m$ - and  $k\mathfrak{S}_n$ -modules respectively. We obtain a  $k\mathfrak{S}_{m+n}$ -module given by  $\text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (V \boxtimes W)$ .

It is easily checked that  $\text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(V \boxtimes W) \cong \text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{m+n}}(W \boxtimes V)$ . We define an outer product by

$$[V] \times [W] = [\text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(V \boxtimes W)],$$

and hence  $[V] \times [W] = [W] \times [V]$ .

Since  $[M(\alpha|\beta)] = [M(\lambda|\mu)]$  when  $\lambda, \mu$  are rearrangements of  $\alpha, \beta$  respectively and  $[M(\alpha|\beta)] \times [M(\gamma|\delta)] = [M(\alpha\#\gamma|\beta\#\delta)]$ , the subset  $\mathcal{Y}(\mathfrak{S})$  of  $\mathcal{R}_k(\mathfrak{S})$  spanned by the isomorphism classes of indecomposable summands of signed Young permutation modules is a graded subring where

$$\mathcal{Y}(\mathfrak{S}) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{Y}(\mathfrak{S}_n)$$

and  $\mathcal{Y}(\mathfrak{S}_n)$  is spanned by  $[M(\alpha|\beta)]$  one for each  $(\alpha|\beta) \in \mathcal{P}^2(n)$ .

**Theorem 2.3.** *Both  $\{[Y(\lambda|p\mu)] : (\lambda|p\mu) \in \mathcal{P}_p^2\}$  and*

$$\mathcal{C} = \{[M(\lambda|p\mu)] : (\lambda|p\mu) \in \mathcal{P}_p^2\}$$

*are  $\mathbb{Z}$ -bases for  $\mathcal{Y}(\mathfrak{S})$  and there is a graded involution  $\varpi : \mathcal{Y}(\mathfrak{S}) \rightarrow \mathcal{Y}(\mathfrak{S})$  defined by  $\varpi([V]) = [V \otimes \text{sgn}(n)]$  for each  $[V] \in \mathcal{Y}(\mathfrak{S}_n)$ .*

**2.4. Listing module.** We refer the reader to [8] for the classical theory of Schur algebras.

Fix  $r \in \mathbb{N}$  and  $\Omega$  be a set. The symmetric group  $\mathfrak{S}_r$  acts on  $\Omega^r$  by place permutation and hence it induces a diagonal action on the set  $\Omega^r \times \Omega^r$ . The set of orbits of the action of  $\mathfrak{S}_r$  on  $\Omega^r \times \Omega^r$  is denoted by  $\mathcal{O}(\Omega^r)$ . Let  $f, g, h \in \mathcal{O}(\Omega^r)$ ,  $(\mathbf{p}, \mathbf{q}) \in f$  and

$$\lambda_{g,h}^f = |\{\mathbf{j} \in \Omega^r : (\mathbf{p}, \mathbf{j}) \in g \text{ and } (\mathbf{j}, \mathbf{q}) \in h\}|.$$

Let  $k(\Omega^r)$  be the formal vector space over  $k$  with a basis  $\{\xi_f : f \in \mathcal{O}(\Omega^r)\}$  and the multiplication be defined by

$$\xi_g \xi_h = \sum_{f \in \mathcal{O}(\Omega^r)} (\lambda_{g,h}^f \cdot 1_k) \xi_f.$$

Notice that  $S(n, r) = k([n]^r)$  is the classical Schur algebra. We denote the algebra  $k(\mathbb{N}^r)$  by  $S(\infty, r)$ . By convention, we set  $S(\infty, 0) = k\xi_0 \cong k$  as algebras. Furthermore, we denote

$$S(\infty) = \bigoplus_{r \in \mathbb{N}_0} S(\infty, r).$$

The algebra  $S(\infty)$  is a bialgebra as follows. Let  $(\mathbf{i}, \mathbf{j}) \in \mathbb{N}^r \times \mathbb{N}^r$  and  $(\mathbf{p}, \mathbf{q}) \in \mathbb{N}^s \times \mathbb{N}^s$  and  $f, g$  be the orbits containing them, respectively. We define the orbit containing  $(\mathbf{i}\#\mathbf{p}, \mathbf{j}\#\mathbf{q}) \in \mathbb{N}^{r+s} \times \mathbb{N}^{r+s}$  as  $f\#g$ . Define the comultiplication  $\Delta : S(\infty) \rightarrow S(\infty) \otimes S(\infty)$  and counit  $\varepsilon : S(\infty) \rightarrow k$  by

$$\Delta(\xi_f) = \sum_{\substack{g \in \mathcal{O}(\mathbb{N}^r), h \in \mathcal{O}(\mathbb{N}^s) \\ f = g\#h}} \xi_g \otimes \xi_h, \quad \varepsilon(\xi_f) = \begin{cases} 1 & f = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If  $V, W$  are  $S(\infty, r)$ - and  $S(\infty, s)$ -modules respectively, then  $V \otimes W$  is an  $S(\infty, r+s)$ -module.

Consider the natural  $S(\infty, 1)$ -module  $E$  where  $E$  is  $k$ -vector space with a basis  $\{e_i : i \in \mathbb{N}\}$  and with the action defined by

$$\xi_f e_j = \begin{cases} e_i & \text{if } f = \{(i, j)\}, \\ 0 & \text{otherwise,} \end{cases}$$

for all  $j \in \mathbb{N}$  and  $f \in \mathcal{O}(\mathbb{N})$ . For each  $r \in \mathbb{N}$ , let  $S^r E$  and  $\bigwedge^r E$  be the symmetric and exterior powers of  $E$  respectively. By convention,  $S^0 E \cong k \cong \bigwedge^0 E$ . For  $\alpha \in \mathcal{C}(n)$ , we set

$$S^\alpha E = S^{\alpha_1} E \otimes \cdots \otimes S^{\alpha_{\ell(\alpha)}} E, \quad \bigwedge^\alpha E = \bigwedge^{\alpha_1} E \otimes \cdots \otimes \bigwedge^{\alpha_{\ell(\alpha)}} E,$$

so that  $S^\alpha E$  and  $\bigwedge^\alpha E$  are  $S(\infty, n)$ -modules. For  $(\alpha|\beta) \in \mathcal{C}^2(n)$ , we define the mixed power the  $S(\infty, n)$ -module  $K^{\alpha|\beta} E := S^\alpha E \otimes \bigwedge^\beta E$ . Notice that  $K^{\delta|\xi} E \cong K^{\alpha|\beta} E$  if  $\delta, \xi$  are rearrangements of  $\alpha, \beta$  respectively. Following Donkin [4, §4], the class of non-isomorphic indecomposable summands of mixed powers for  $S(\infty, n)$  are parametrised by the set  $\mathcal{P}_p^2(n)$ . They are called listing modules denoted by  $\text{List}^{\lambda|p\mu} E$  such that, for each  $(\lambda|p\mu) \in \mathcal{P}_p^2(n)$ ,  $\text{List}^{\lambda|p\mu} E$  is a summand of  $K^{\lambda|p\mu} E$  with multiplicity one and any other summand of  $K^{\lambda|p\mu} E$  are of the form  $\text{List}^{\delta|p\theta} E$  where  $(\delta|p\theta) \in \mathcal{P}_p^2(n)$  and  $(\delta|p\theta) \triangleright (\lambda|p\mu)$ . Furthermore, under the Schur functor  $f$ , we have  $f(K^{\alpha|\beta} E) \cong M(\alpha|\beta)$  and  $f(\text{List}^{\lambda|p\mu} E) \cong Y(\lambda|p\mu)$ .

Let  $\mathcal{L}(S)$  be the graded subring of the representation ring of  $S(\infty)$  generated by  $[S^n E]$  and  $[\bigwedge^n E]$  for every  $n \in \mathbb{N}_0$  with the gradation

$$\mathcal{L}(S) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{L}(S(\infty, n))$$

where  $\mathcal{L}(S(\infty, n))$  is spanned by the mixed powers  $[K^{\alpha|\beta} E]$  such that  $(\alpha|\beta) \in \mathcal{C}^2(n)$ .

**Theorem 2.4.** *Both  $\{[\text{List}^{\lambda|p\mu} E] : (\lambda|p\mu) \in \mathcal{P}_p^2\}$  and*

$$\mathcal{D} = \{[K^{\lambda|p\mu} E] : (\lambda|p\mu) \in \mathcal{P}_p^2\}$$

*are  $\mathbb{Z}$ -bases for  $\mathcal{L}(S)$ . Furthermore, the ring  $\mathcal{L}(S)$  has a graded involution  $\omega : \mathcal{L}(S) \rightarrow \mathcal{L}(S)$  defined by  $\omega(S^n E) = \bigwedge^n E$ .*

The following categorification theorem of the rings  $\mathcal{V}(\mathfrak{S})$  and  $\mathcal{L}(S)$  is essential throughout this article.

**Theorem 2.5** ([4, §4.1(10,13)]). *We have a commutative diagram*

$$\begin{array}{ccc}
 \mathcal{L}(S) & \xrightarrow{f} & \mathcal{Y}(\mathfrak{S}) \\
 \uparrow \omega & \swarrow \theta & \nearrow \phi \\
 & \Gamma^{(p)} & \\
 \uparrow \omega & \swarrow \theta & \nearrow \phi \\
 \mathcal{L}(S) & \xrightarrow{f} & \mathcal{Y}(\mathfrak{S}) \\
 & \nwarrow \psi & \nearrow \phi \\
 & \Gamma^{(p)} & 
 \end{array}$$

where the maps  $f, \theta, \phi$  are isomorphisms of graded rings where, for each  $(\alpha|\beta) \in \mathcal{C}^2$ ,  $f([K^{\alpha|\beta}E]) = [M(\alpha|\beta)]$ ,  $\theta(h_\alpha e_\beta) = [K^{\alpha|\beta}E]$  and  $\phi(h_\alpha e_\beta) = [M(\alpha|\beta)]$ .

We end this section with a remark.

**Remark 2.6.** Let  $\mathcal{M}$  be the Mullineux map on  $p$ -restricted partitions (see, for example, [1, §6]) and let  $\lambda(0)$  be the  $p$ -restricted partition such that  $\lambda = \lambda(0) + p\xi$  for some partition  $\xi$  (here,  $\lambda(0) + p\xi$  has the obvious meaning componentwise summation of partitions). Using Theorem 2.5 and [2, Theorem 3.21], for each  $(\lambda|p\mu) \in \mathcal{P}_p^2$ , since  $\varpi([Y(\lambda|p\mu)]) = [Y(\mathcal{M}(\lambda(0)) + p\mu|\lambda - \lambda(0))]$ , we have

$$\omega([\text{List}^{\lambda|p\mu}E]) = [\text{List}^{\mathcal{M}(\lambda(0)) + p\mu|\lambda - \lambda(0)}E].$$

Similarly, in  $\Gamma^{(p)}$ , we have

$$\psi(h_\lambda e_{p\mu}) = h_{\mathcal{M}(\lambda(0)) + p\mu} e_{\lambda - \lambda(0)}.$$

### 3. THE RING $\Gamma^{(m)}$

Throughout this section, we fix  $m \in \mathbb{N}$ . In this section, we derive some properties about the graded ring  $\Gamma^{(m)}$ . Recall that  $\Gamma^{(m)}$  is the quotient ring  $\langle \Lambda(X), \Lambda(Y) \rangle / I$  where  $I$  is the ideal generated by  $\sum_{i=0}^d (-1)^i h_i(X) e_{d-i}(Y)$  for every positive integer  $d$  such that  $m \nmid d$ . Furthermore,  $\Gamma^{(m)}$  is the polynomial ring on  $\{h_i, e_{jm} : i, j \in \mathbb{N}\}$  where  $h_n = h_n(X) + I$  and  $e_n = e_n(Y) + I$ , and it has a  $\mathbb{Z}$ -basis

$$\mathcal{B} = \{h_\lambda e_{m\mu} : (\lambda|m\mu) \in \mathcal{P}_m^2\}.$$

We begin with a lemma.

**Lemma 3.1.**

- (i) For each  $d \in \mathbb{N}$ ,  $e_d$  is a  $\mathbb{Z}$ -linear combination of  $h_\lambda e_{sm}$  for some partitions  $\lambda$  and  $s \in \mathbb{N}_0$  such that  $|\lambda| + sm = d$ .
- (ii) Let

$$a_{ij} = \begin{cases} h_{1-i+j} & \text{if } m \nmid j, \\ (-1)^{j+1} e_j & \text{if } j = sm \text{ and } i = 1, \\ 1 & \text{if } j = sm \text{ and } i = sm + 1, \\ 0 & \text{otherwise.} \end{cases}$$



Then, for each  $d \geq 1$ , we have  $e_d = \det(a_{ij})_{1 \leq i, j \leq d}$ .  
 (iii) If we set  $x_i = y_i$  for each  $i \in \mathbb{N}$  then  $I = 0$  and  $\Gamma^{(m)} = \Lambda(X)$  where  $h_\lambda = h_\lambda(X)$  and  $e_\mu = e_\mu(X)$ .

*Proof.* The proof of part (i) is straightforward by using induction on  $d$  and the relation  $\sum_{i=1}^d (-1)^i h_i e_{d-i} = 0$  when  $m \nmid d$ . Alternatively, it follows from part (ii).

For part (ii), let  $A_d = (a_{ij})_{1 \leq i, j \leq d}$ . Notice that  $A_1 = (e_1)$  and hence  $e_1 = \det(A_1)$ . Fix a  $d \in \mathbb{N}$ . By induction, suppose that  $e_j = \det(A_j)$  for all  $1 \leq j \leq d-1$ . Let  $w_d$  be the  $((d-1) \times 1)$ -column matrix with the first entry  $(-1)^{d+1} e_d$  and 0 elsewhere, let  $z_i$  be the  $(i \times 1)$ -column matrix whose entries are  $h_d, h_{d-1}, \dots, h_{d-i+1}$  read from the top and let  $v_j$  be the  $(1 \times j)$ -row matrix whose last entry is 1 and 0 elsewhere. If  $d = \ell m$  then, expand along the last column,

$$\det(A_d) = \det \begin{pmatrix} A_{d-1} & w_d \\ v_{d-1} & 0 \end{pmatrix} = (-1)^{\ell m+1} \cdot (-1)^{\ell m+1} e_{\ell m} = e_d.$$

Suppose that  $m \nmid d$ . In particular,  $m \neq 1$  and thus  $h_1 = e_1$ . Let  $B_s = \begin{pmatrix} A_{d-s} & z_{d-s} \\ v_{d-s} & h_s \end{pmatrix}$ . We claim that, for all  $1 \leq s \leq d-1$ ,  $\det(B_s) = \sum_{i=s}^d (-1)^{i-s} h_i e_{d-i}$ . When  $s = d-1$ , we have

$$\det(B_{d-1}) = \det \begin{pmatrix} A_1 & z_1 \\ v_1 & h_{d-1} \end{pmatrix} = e_1 h_{d-1} - h_d = e_1 h_{d-1} - e_0 h_d.$$

Expand  $\det(B_s)$  along the last row, by induction, we have

$$\begin{aligned} \det(B_s) &= h_s \det(A_{d-s}) - \det(B_{s+1}) = h_s e_{d-s} - \sum_{i=s+1}^d (-1)^{i-(s+1)} h_i e_{d-i} \\ &= \sum_{i=s}^d (-1)^{i-s} h_i e_{d-i}. \end{aligned}$$

Notice that  $A_d = B_1$ . Therefore, we conclude that

$$\det(A_d) = \det(B_1) = \sum_{i=1}^d (-1)^{i-1} h_i e_{d-i} = e_d.$$

For part (iii), if  $x_i = y_i$  for each  $i \in \mathbb{N}$ , since  $\sum_{i=0}^d (-1)^i h_i(X) e_{d-i}(X) = 0$  for all  $d$  by Theorem 2.1(ii), we have  $I = 0$ . So  $h_i = h_i(X)$ ,  $e_i = e_i(X)$  and  $\Gamma^{(m)} = \Lambda(X)$ .  $\square$

In view of Lemma 3.1(i), we introduce the following notation. For each  $n \in \mathbb{N}$ , let

$$e_n = \sum_{n=|\mu|+sm} d_{n,\mu} h_\mu e_{sm}$$

where  $d_{n,\mu} \in \mathbb{Z}$  with respect to the basis  $\mathcal{B}$  so that  $s \in \mathbb{N}_0$  in the above summation. Notice that if  $n = \ell m$  for some  $\ell$  then  $d_{\ell m, \mu} = 0$  for all  $\mu \neq \emptyset$  and  $d_{\ell m, \emptyset} = 1$ .

**Corollary 3.2.** *Let  $\mu \in \mathcal{P}(n)$ . For all  $r \in \mathbb{N}_0$ , we have*

$$d_{n+rm, \mu} = d_{n, \mu}.$$

*Proof.* For each  $(\alpha|m\beta) \in \mathcal{P}_m^2$ , we denote the coefficient of  $h_\alpha e_{m\beta}$  in  $z \in \Gamma^{(m)}$  as  $(z, h_\alpha e_{m\beta})$ . Then  $d_{n,\mu} = (e_n, h_\mu)$  and  $d_{n+rm,\mu} = (e_{n+rm}, h_\mu e_{rm})$ . Let  $A_d = (a_{ij})_{1 \leq i,j \leq d}$  be defined as in Lemma 3.1(ii) and, by which, we have both  $e_n = \det(A_n)$  and  $e_{n+rm} = \det(A_{n+rm})$ . By virtue of the entries of  $A_d$ , an element of the form  $h_\alpha e_{jm}$  only appears in  $e_{jm}(A_d)^{(1,jm)}$  where  $(A_d)^{(1,jm)}$  is the  $(1, jm)$ -minor of the matrix  $A_d$ .

Let  $C_d$  be the matrix obtained from  $A_d$  by replacing all entries of the form  $(-1)^{\ell+1}e_\ell$  by 0. Expand along the  $rm$ th column of  $A_n$ , we have

$$\begin{aligned} (e_{n+rm}, h_\mu e_{rm}) &= (\det(A_{n+rm}), h_\mu e_{rm}) \\ &= (e_{rm}(A_{n+rm})^{(1,rm)}, h_\mu e_{rm}) \\ &= \left( e_{rm} \det \begin{pmatrix} J & B \\ \mathbf{0} & C_n \end{pmatrix}, h_\mu e_{rm} \right) \\ &= (\det(C_n), h_\mu) \end{aligned}$$

where  $J$  is an upper unitriangular matrix and  $B$  is some suitable matrix. By similar calculation, we have  $(e_n, h_\mu) = (\det(C_n), h_\mu)$ . So we conclude that  $d_{n+rm,\mu} = d_{n,\mu}$ .  $\square$

In view of Corollary 3.2, we have the following notation.

**Notation 3.3.** Let  $n \in \mathbb{N}_0$ . We denote

$$\mathcal{P}(n; m) = \{\mu \in \mathcal{P} : n = |\mu| + sm \text{ for some } s \in \mathbb{N}_0\}.$$

For each partition  $\mu$ , we write  $d_\mu = d_{n,\mu}$  for the common number for all  $n = |\mu| + rm$  where  $r \in \mathbb{N}_0$  so that, for all  $n \in \mathbb{N}_0$ , we have

$$e_n = \sum_{n=|\mu|+sm} d_\mu h_\mu e_{sm} = \sum_{\mu \in \mathcal{P}(n; m)} d_\mu h_\mu e_{n-|\mu|}.$$

#### 4. THE STRAIGHTENING RULE

Recall the  $\mathbb{Z}$ -bases  $\mathcal{B}$ ,  $\mathcal{C}$  and  $\mathcal{D}$  of the rings  $\Gamma^{(m)}$ ,  $\mathcal{Y}(\mathfrak{S})$  and  $\mathcal{L}(\mathfrak{S})$  denoted in Theorems 2.2, 2.3 and 2.4 respectively. Our main result in this section provides an explicit formula to write, for each  $(\alpha|\beta) \in \mathcal{C}^2(n)$ , the element  $h_\alpha e_\beta$  in terms of  $\mathcal{B}$ . As a consequence, we have the explicit formulae to write  $[M(\alpha|\beta)]$  and  $[K^{\alpha|\beta}E]$  as  $\mathbb{Z}$ -linear combinations in terms of the bases  $\mathcal{C}$  and  $\mathcal{D}$  respectively.

We begin with a list of notation.

**Notation 4.1.** Fix a positive integer  $m$ , let  $\lambda \in \mathcal{P}(n)$  and let  $b = \lfloor \frac{n}{m} \rfloor$ .

- (i) Let  $\mathcal{C}(\lambda)$  be the set of compositions of the form  $\delta = (\delta_1, \dots, \delta_{\ell(\lambda)})$  such that its rearrangement is  $\lambda$  and let  $c_\lambda = |\mathcal{C}(\lambda)|$ , i.e.,

$$c_\lambda = \frac{n!}{\prod_{i \in \mathbb{N}} n_i!}$$

where  $n_i$  is the number of parts of  $\lambda$  of size  $i$ .

- (ii) Let  $c_\lambda^{(m)}$  be the number of compositions  $\delta = (\delta_1, \dots, \delta_{\ell(\lambda)}) \in \mathcal{C}(\lambda)$  such that
- $$\delta_1 + \delta_2 + \dots + \delta_j \not\equiv 0 \pmod{m}$$
- for all  $1 \leq j \leq \ell(\lambda)$ .
- (iii) Let  $\varepsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}$ .
- (iv) For a finite sequence  $A = (\delta^{(1)}, \dots, \delta^{(r)})$  of partitions, we write  $c_A = \prod_{i=1}^r c_{\delta^{(i)}}$  and  $\ell(A) = r$ . Similarly, we write  $c_A^{(m)} = \prod_{i=1}^r c_{\delta^{(i)}}^{(m)}$  and  $\varepsilon_A = \prod_{i=1}^r \varepsilon_{\delta^{(i)}}$ .
- (v) If  $\delta, \xi$  are compositions such that  $\delta \# \xi$  can be rearranged to  $\lambda$ , we write  $\lambda = \delta \cup \xi$  (or  $\lambda = \xi \cup \delta$ ). The notation  $\lambda = \xi^{(1)} \cup \dots \cup \xi^{(r)}$  has the obvious meaning.
- (vi) Let  $W(\lambda)$  be the set consists of sequences  $(\delta^{(1)}, \dots, \delta^{(r)}, \xi)$  of partitions for some  $r \in \mathbb{N}_0$  such that  $\xi \cup \delta^{(1)} \cup \dots \cup \delta^{(r)} = \lambda$ ,  $\delta^{(i)} \neq \emptyset$  and  $m \mid |\delta^{(i)}|$  for all  $1 \leq i \leq r$ .
- (vii) For each  $1 \leq i \leq b$ , let  $W_i(\lambda) = \{A \in W(\lambda) : \ell(A) = i + 1\}$ .
- (viii) For each  $1 \leq i \leq b$ , let  $P_i$  be the subset of  $\mathcal{C}(\lambda)$  consisting of all compositions  $(\mu_1, \dots, \mu_{\ell(\lambda)})$  such that  $\mu_1 + \mu_2 + \dots + \mu_r = mi$  for some  $1 \leq r \leq \ell(\lambda)$ .
- (ix) Let  $\xi$  be a partition and  $\beta$  be a composition. Suppose that  $s = \ell(\beta)$ . We denote by  $V(\beta; (\xi, m\mu))$  the set consisting of the  $s$ -tuples

$$(\xi^{(1)}, \dots, \xi^{(s)}) \in \mathcal{P}(\beta_1; m) \times \dots \times \mathcal{P}(\beta_s; m)$$

such that  $\xi = \xi^{(1)} \cup \dots \cup \xi^{(s)}$  and  $m\mu = (\beta_1 - |\xi^{(1)}|) \cup \dots \cup (\beta_s - |\xi^{(s)}|)$ . Furthermore, we write

$$c_{\beta; (\xi, m\mu)}^{(m)} = \sum_{A \in V(\beta; (\xi, m\mu))} \varepsilon_A c_A^{(m)}.$$

By convention,  $c_\emptyset = 1$ ,  $c_\emptyset^{(m)} = 1$ ,  $\varepsilon_\emptyset = 1$ ,  $W(\emptyset) = \{(\emptyset)\}$  and  $c_{\beta; (\xi, m\mu)}^{(m)} = 0$  if  $V(\beta; (\xi, m\mu)) = \emptyset$ .

For example, let  $\lambda = (3, 2, 1, 1)$  and  $m = 3$ . Then the compositions  $(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathcal{C}(\lambda)$  with the property that  $3 \nmid \sum_{i=1}^j \mu_i$  for all  $1 \leq j \leq 4$  are

$$(1, 1, 3, 2), \quad (1, 3, 1, 2), \quad (1, 1, 2, 3).$$

Thus  $c_{(3,2,1,1)}^{(3)} = 3$ . Furthermore,

$$W(\lambda) = \{(\lambda), ((3), (2, 1, 1)), ((2, 1), (3, 1)), ((3), (2, 1), (1)), ((2, 1), (3), (1))\}.$$

For another example, if  $\lambda \neq \emptyset$  and  $m \mid |\lambda|$ , then  $c_\lambda^{(m)} = 0$ . Also,

$$W((m)) = \{((m)), ((m), \emptyset)\}.$$

We may now state our main theorem.

**Theorem 4.2.** *Fix a positive integer  $m$ . For each  $(\alpha|\beta) \in \mathcal{C}^2(n)$ ,*

$$h_\alpha e_\beta = \sum c_{\beta; (\xi, m\mu)}^{(m)} h_\lambda e_{m\mu}$$

where the sum runs over all  $(\lambda|m\mu) \in \mathcal{P}_m^2(n)$  such that  $\lambda = \alpha \cup \xi$  for some partition  $\xi$ . In particular, we have  $d_\lambda = \varepsilon_\lambda c_\lambda^{(m)}$  for each partition  $\lambda$ .

We illustrate the theorem with an example. Let  $m = 3$ . Then

$$e_2 = \sum_{2=|\lambda|+3s} \varepsilon_\lambda c_\lambda^{(3)} h_\lambda e_{3s} = -h_{(2)} + h_{(1,1)}$$

since  $c_{(2)}^{(3)} = 1$ ,  $\varepsilon_{(2)} = -1$ ,  $c_{(1,1)}^{(3)} = 1$  and  $\varepsilon_{(1,1)} = 1$ . Similarly, we have

$$e_4 = -h_{(4)} + h_{(3,1)} + h_{(2,2)} - h_{(2,1,1)} + h_{(1)}e_{(3)}.$$

So, for example, we get

$$\begin{aligned} h_{(5)}e_{(4,3,2)} &= h_{(5)} \left( -h_{(4)} + h_{(3,1)} + h_{(2,2)} - h_{(2,1,1)} + h_{(1)}e_{(3)} \right) e_{(3)} \left( -h_{(2)} + h_{(1,1)} \right) \\ &= h_{5,4,2}e_3 - h_{5,4,1,1}e_3 - h_{5,3,2,1}e_3 + h_{5,3,1,1,1}e_3 - h_{5,2,2,2}e_3 + 2h_{5,2,2,1,1}e_3 \\ &\quad - h_{5,2,1,1,1,1}e_3 - h_{5,2,1}e_{3,3} + h_{5,1,1,1}e_{3,3}, \end{aligned}$$

here, after the last equality, we have deliberately removed the parentheses.

To prove Theorem 4.2, we need the following two lemmas.

**Lemma 4.3.** *For each partition  $\lambda$ ,*

$$d_\lambda = -\varepsilon_\lambda \left( \sum_{A \in W(\lambda)} (-1)^{\ell(A)} c_A \right).$$

*Proof.* Let  $b = \lfloor \frac{n}{m} \rfloor$ . Recall that the elementary symmetric functions can be written in terms of the complete symmetric functions as

$$e_n(X) = \sum_{\lambda \in \mathcal{P}(n)} \varepsilon_\lambda c_\lambda h_\lambda(X)$$

with respect to the set of variables  $X = \{x_i : i \in \mathbb{N}\}$  (see [13, Chapter I.2 Example 20]). Similarly,  $e_n(Y) = \sum_{\lambda \in \mathcal{P}(n)} \varepsilon_\lambda c_\lambda h_\lambda(Y)$ . Let  $x_i = y_i$  for each  $i \in \mathbb{N}$ . Using Lemma 3.1(iii) and Notation 3.3, we have

$$\sum_{\mu \in \mathcal{P}(n;m)} d_\mu h_\mu(X) \left( \sum_{\delta \in \mathcal{P}(n-|\mu|)} \varepsilon_\delta c_\delta h_\delta(X) \right) = \sum_{\lambda \in \mathcal{P}(n)} \varepsilon_\lambda c_\lambda h_\lambda(X).$$

Fix a partition  $\lambda \in \mathcal{P}(n)$ . If  $\lambda = \mu \cup \delta$  with  $\mu, \delta \in \mathcal{P}$  then  $\mu$  uniquely determines  $\delta$  and vice versa. By Theorem 2.1(i), comparing the coefficients of  $h_\lambda(X)$ , we have

$$(1) \quad \varepsilon_\lambda c_\lambda = \sum_{i=0}^b \sum_{\substack{\delta \in \mathcal{P}(mi), \\ \delta \cup \mu = \lambda}} \varepsilon_\delta d_\mu c_\delta.$$

We prove the equality in the statement by using induction on the size of  $\lambda$ . If  $\lambda = \emptyset$  then  $d_\emptyset = 1$  and  $-\varepsilon_\emptyset((-1)^{\ell((\emptyset))})c_\emptyset = 1$ . Using Equation 1 and induction, we have

$$\begin{aligned} \varepsilon_\lambda c_\lambda &= d_\lambda + \sum_{i=1}^b \sum_{\substack{\delta \in \mathcal{P}(mi), \\ \delta \cup \mu = \lambda}} \varepsilon_\delta d_\mu c_\delta = d_\lambda - \sum_{i=1}^b \sum_{\substack{\delta \in \mathcal{P}(mi), \\ \delta \cup \mu = \lambda}} \varepsilon_\delta \varepsilon_\mu c_\delta \left( \sum_{A \in W(\mu)} (-1)^{\ell(B)} c_B \right) \\ &= d_\lambda + \varepsilon_\lambda \sum_{i=1}^b \sum_{\substack{\delta \in \mathcal{P}(mi), \\ \delta \cup \mu = \lambda}} \left( \sum_{A \in W(\mu)} (-1)^{\ell(B)+1} c_\delta c_B \right) \end{aligned}$$

Notice that  $W(\lambda) \setminus \{(\lambda)\}$  consists of sequences of partitions, one for each  $i \in [b]$ ,  $(\delta, \delta^{(1)}, \dots, \delta^{(j)}, \xi)$  such that  $\delta \cup \mu = \lambda$ ,  $|\delta| = im$  and  $(\delta^{(1)}, \dots, \delta^{(j)}, \xi) \in W(\mu)$ . So we get

$$d_\lambda = \varepsilon_\lambda \left( c_\lambda - \sum_{A \in W(\lambda) \setminus \{(\lambda)\}} (-1)^{\ell(A)} c_A \right) = -\varepsilon_\lambda \left( \sum_{A \in W(\lambda)} (-1)^{\ell(A)} c_A \right).$$

The proof is now complete.  $\square$

**Lemma 4.4.** *Let  $\lambda \in \mathcal{P}(n)$  and  $b = \lfloor \frac{n}{m} \rfloor$ . For each  $j \in [b]$ , have*

$$\sum_{1 \leq i_1 < \dots < i_j \leq b} |P_{i_1} \cap \dots \cap P_{i_j}| = \sum_{A \in W_j(\lambda)} c_A.$$

*Proof.* In the proof, we denote the disjoint union of sets  $S_1, \dots, S_a$  by  $\bigsqcup_{i=1}^a S_i$ . Let

$$\psi : \bigsqcup_{1 \leq i_1 < \dots < i_j \leq b} P_{i_1} \cap \dots \cap P_{i_j} \rightarrow \bigsqcup_{(\delta^{(1)}, \dots, \delta^{(j)}, \xi) \in W_j(\lambda)} \mathcal{C}(\delta^{(1)}) \times \dots \times \mathcal{C}(\delta^{(j)}) \times \mathcal{C}(\xi)$$

be defined as follows. For each  $\eta \in P_{i_1} \cap \dots \cap P_{i_j}$ , we have positive integers  $1 \leq s_1 < \dots < s_j \leq \ell(\lambda)$  such that  $\sum_{t=1}^{s_r} \eta_t = mi_r$  for each  $1 \leq r \leq j$ . Therefore, for each  $1 \leq t \leq j+1$ ,

$$\eta^{(t)} = (\eta_{s_{t-1}+1}, \eta_{s_{t-1}+2}, \dots, \eta_{s_t}) \in \mathcal{C}(\delta^{(t)})$$

for some partition  $\delta^{(t)}$  of size  $m(i_t - i_{t-1})$  where  $i_0 = 0 = s_0$  and  $s_{j+1} = \ell(\lambda)$ . Then

$$\psi(\eta) = \prod_{t=1}^{j+1} \eta^{(t)} \in \mathcal{C}(\delta^{(1)}) \times \dots \times \mathcal{C}(\delta^{(j)}) \times \mathcal{C}(\xi)$$

since  $(\delta^{(1)}, \dots, \delta^{(j)}, \xi) \in W_j(\lambda)$ . The map  $\psi$  is invertible with the inverse  $\phi$  given as follows. For each  $(\delta^{(1)}, \dots, \delta^{(j)}, \xi) \in W_j(\lambda)$  and

$$h = (\eta^{(1)}, \dots, \eta^{(j)}, \eta^{(j+1)}) \in \mathcal{C}(\delta^{(1)}) \times \dots \times \mathcal{C}(\delta^{(j)}) \times \mathcal{C}(\xi),$$

we define  $\phi(h) = \eta^{(1)} \# \dots \# \eta^{(j)} \# \eta^{(j+1)} \in P_{i_1} \cap \dots \cap P_{i_j}$  where  $mi_r = \sum_{t=1}^r |\eta^{(t)}|$  for  $1 \leq r \leq j$ . The bijection  $\psi$  shows that

$$\sum_{1 \leq i_1 < \dots < i_j \leq b} |P_{i_1} \cap \dots \cap P_{i_j}| = \sum_{(\delta^{(1)}, \dots, \delta^{(j)}, \xi) \in W_j(\lambda)} |\mathcal{C}(\delta^{(1)}) \times \dots \times \mathcal{C}(\delta^{(j)}) \times \mathcal{C}(\xi)| = \sum_{A \in W_j(\lambda)} c_A.$$

□

We may now prove our main theorem.

*Proof of Theorem 4.2.* Let  $b = \lfloor \frac{n}{m} \rfloor$ . By the inclusion-exclusion principle and Lemmas 4.3 and 4.4, we have

$$\begin{aligned}
c_\lambda^{(m)} &= \left| \bigcap_{i=1}^b (\mathcal{C}(\lambda) \setminus P_i) \right| = |\mathcal{C}(\lambda)| - \sum_{j=1}^b (-1)^{j+1} \left( \sum_{1 \leq i_1 < \dots < i_j \leq b} |P_{i_1} \cap \dots \cap P_{i_j}| \right) \\
&= |\mathcal{C}(\lambda)| - \sum_{j=1}^b (-1)^{j+1} \left( \sum_{A \in W_j(\lambda)} c_A \right) \\
&= - \left( \sum_{A \in W(\lambda)} (-1)^{\ell(A)} c_A \right) \\
&= \varepsilon_\lambda d_\lambda,
\end{aligned}$$

i.e.,  $d_\lambda = \varepsilon_\lambda c_\lambda^{(m)}$ . This proves the particular case. Let  $s = \ell(\beta)$ . For the general case, for each  $1 \leq i \leq s$ , we have  $e_{\beta_i} = \sum_{\xi^{(i)} \in \mathcal{P}(\beta_i; m)} \varepsilon_{\xi^{(i)}} c_{\xi^{(i)}}^{(m)} h_{\xi^{(i)}} e_{\beta_i - |\xi^{(i)}|}$ . So

$$e_\beta = \prod_{i=1}^s e_{\beta_i} = \sum_{(\xi|m\mu) \in \mathcal{P}_m^2(|\beta|)} c_{\beta;(\xi, m\mu)}^{(m)} h_\xi e_{m\mu}.$$

Therefore,

$$h_\alpha e_\beta = \sum_{(\xi|m\mu) \in \mathcal{P}_m^2(|\beta|)} c_{\beta;(\xi, m\mu)}^{(m)} h_{\alpha \cup \xi} e_{m\mu} = \sum_{\substack{(\lambda|m\mu) \in \mathcal{P}_m^2(n), \\ \lambda = \xi \cup \alpha, \xi \in \mathcal{P}}} c_{\beta;(\xi, m\mu)}^{(m)} h_\lambda e_{m\mu}.$$

□

Using Theorems 2.5 and 4.2, we obtain the following immediate corollaries.

**Corollary 4.5.** *Let  $(\alpha|\beta) \in \mathcal{C}^2(n)$ . Then*

- (i)  $[M(\alpha|\beta)] = \sum c_{\beta;(\xi, p\mu)}^{(p)} [M(\lambda|p\mu)]$ , and
- (ii)  $[K^{\alpha|\beta} E] = \sum c_{\beta;(\xi, p\mu)}^{(p)} [K^{\lambda|p\mu} E]$

where both of the sums above run over all  $(\lambda|p\mu) \in \mathcal{P}_p^2(n)$  such that  $\lambda = \alpha \cup \xi$  for some partition  $\xi$ .

We demonstrate Corollary 4.5 with an example. When  $p = 3$ , as we have computed earlier, using the maps  $\theta$  and  $\phi$  as in Theorem 2.5, we have

$$\begin{aligned} [M(\emptyset|(2))] &= -[M((2)|\emptyset)] + [M((1, 1)|\emptyset)], \\ [M(\emptyset|(4))] &= -[M((4)|\emptyset)] + [M((3, 1)|\emptyset)] - [M((2, 1, 1)|\emptyset)] + [M((1)|(3))], \\ [\bigwedge^2 E] &= -[S^2 E] + [S^{(1, 1)} E], \\ [\bigwedge^4 E] &= -[S^4 E] + [S^{(3, 1)} E] - [S^{(2, 1, 1)} E] + [E \otimes \bigwedge^3 E]. \end{aligned}$$

Let  $(M(\alpha|\beta) : Y(\delta|p\xi))$  denote the multiplicity of  $Y(\delta|p\xi)$  as a direct summand of  $M(\alpha|\beta)$ . Similarly, we have the multiplicity  $(K^{\alpha|\beta} E : \text{List}^{\delta|p\xi} E)$ . The following corollary is clear.

**Corollary 4.6.** *Let  $(\alpha|\beta) \in \mathcal{C}^2(n)$  and  $(\delta|p\xi) \in \mathcal{P}_p^2(n)$ . Then*

- (i)  $(M(\alpha|\beta) : Y(\delta|p\xi)) = \sum c_{\beta;(\xi, p\mu)}^{(p)} (M(\lambda|p\mu) : Y(\delta|p\xi))$ , and
- (ii)  $(K^{\alpha|\beta} E : \text{List}^{\delta|p\xi} E) = \sum c_{\beta;(\xi, p\mu)}^{(p)} (K^{\lambda|p\mu} E : \text{List}^{\delta|p\xi} E)$

where both of the sums above run over all  $(\lambda|p\mu) \in \mathcal{P}_p^2(n)$  such that  $\lambda = \alpha \cup \xi$  for some partition  $\xi$ .

Recall that  $Y(\lambda|p\mu)$  (respectively,  $\text{List}^{\lambda|p\mu} E$ ) is a summand of  $M(\lambda|p\mu)$  (respectively,  $K^{\lambda|p\mu} E$ ) with multiplicity one and any other summand  $Y(\alpha|p\beta)$  (respectively,  $\text{List}^{\alpha|p\beta} E$ ) necessarily satisfies  $(\alpha|p\beta) \triangleright (\lambda|p\mu)$ . Applying our result, we have a similar description when  $(\lambda|p\mu)$  is replaced by a general pair of compositions  $(\alpha|\beta)$ .

**Corollary 4.7.** *Let  $(\alpha|\beta) \in \mathcal{C}^2(n)$  and  $s = \ell(\beta)$ . For each  $1 \leq i \leq s$ , let  $\beta_i = p\eta_i + r_i$  such that  $0 \leq r_i \leq p - 1$  and let  $r = \sum_{i=1}^s r_i$ . Suppose that  $\alpha \# (1^r)$  and  $(\eta_1, \dots, \eta_s)$  are rearranged to the partitions  $\lambda$  and  $\mu$  respectively.*

- (i) *The signed Young permutation module  $M(\alpha|\beta)$  has the summand  $Y(\lambda|p\mu)$  such that  $(M(\alpha|\beta) : Y(\lambda|p\mu)) = 1$  and, if  $Y(\delta|p\theta)$  is a summand of  $M(\alpha|\beta)$ , then  $(\delta|p\theta) \supseteq (\lambda|p\mu)$ .*
- (ii) *The mixed power  $K^{\alpha|\beta} E$  has the summand  $\text{List}^{\lambda|p\mu} E$  such that  $(K^{\alpha|\beta} E : \text{List}^{\lambda|p\mu} E) = 1$  and, if  $\text{List}^{\delta|p\theta} E$  is a summand of  $K^{\alpha|\beta} E$ , then  $(\delta|p\theta) \supseteq (\lambda|p\mu)$ .*

*Proof.* Since  $M(\alpha|\beta) \cong M(\zeta|\nu)$  if  $\zeta, \nu$  are rearrangements of  $\alpha, \beta$  respectively, we may assume that  $\alpha, \beta$  are partitions so that  $\lambda = \alpha \# (1^r)$  and  $\mu = (\eta_1, \dots, \eta_s)$  in the statement. For each  $0 \leq d \leq p - 1$ , we have  $\varepsilon_{(1^d)} c_{(1^d)}^{(p)} = 1$ . It is easy to check that

$$V(\beta; ((1^{|\beta| - p|\mu|}), p\mu)) = \{((1^{r_1}), \dots, (1^{r_s}))\}$$

and hence  $c_{\beta;((1^{|\beta| - p|\mu|}), p\mu)}^{(p)} = 1$ . If  $c_{\beta;(\xi, p\theta)}^{(p)} \neq 0$  as in Corollary 4.5(i), then  $\delta = \alpha \cup \xi$ ,  $|\xi| \geq r$  and  $\theta_i \leq \mu_i$  for all  $i$ . Suppose that  $Y(\lambda|p\mu)$  is a summand of  $M(\delta|p\theta)$ . We have  $(\lambda|p\mu) \supseteq (\delta|p\theta)$ . In particular,

$$|\alpha| + r = |\lambda| \geq |\delta| = |\alpha| + |\xi|.$$

Hence  $|\xi| = r$ ,  $|\lambda| = |\delta|$  and  $\alpha\#(1^r) = \lambda \supseteq \delta = \alpha \cup \xi$  where, here,  $\supseteq$  is the usual dominance order on  $\mathcal{P}(n)$ . Therefore,  $\xi = (1^r)$ . Furthermore,  $p|\theta| = p|\mu|$  and hence we conclude that  $\theta = \mu$ . This shows that any summand  $Y(\delta|p\theta)$  of  $M(\alpha|\beta)$  satisfies  $(\delta|p\theta) \supseteq (\lambda|p\mu)$  and  $(M(\alpha|\beta) : Y(\lambda|p\mu)) = 1$ . Part (ii) can be proved in the similar fashion or follows from part (i) using Theorem 2.5.  $\square$

In [6, Proposition 7.1], Giannelli, O'Donovan, Wildon and the author found the label of signed Young module when  $M(\alpha|\beta)$  is indecomposable in the odd characteristics case, i.e., the pair of partitions  $(\delta|p\theta)$  such that  $M(\alpha|\beta) \cong Y(\delta|p\theta)$ . Direct application of Corollary 4.7 gives an alternative proof.

**Corollary 4.8** ([6, Proposition 7.1]). *Let  $a, b \in \mathbb{N}_0$ ,  $a \neq 0$  and  $b = sp + r$  where  $0 \leq r \leq p - 1$ . Then  $M(\emptyset|(b)) \cong Y((1^r)|p(s))$  and, if  $p \mid a + b$ ,  $M((a)|(b)) \cong Y((a, 1^r)|p(s))$ .*

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